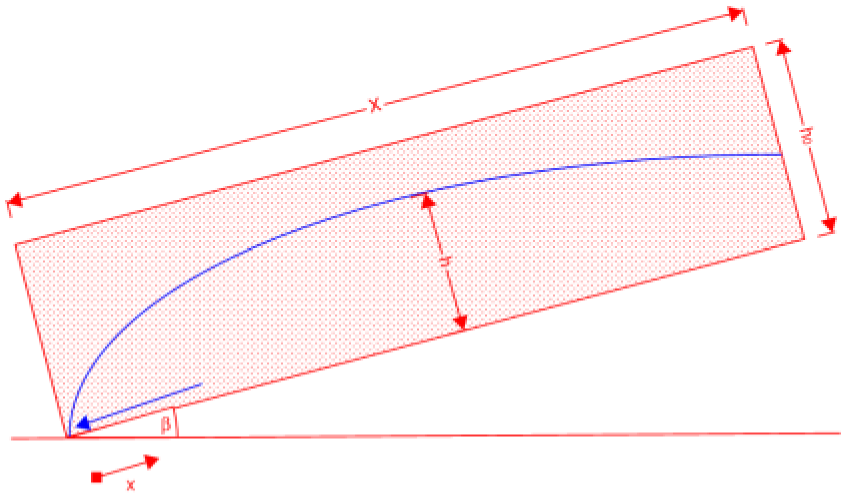
Boussinesq equation

Based on the Dupuit approximation some modifications can be made for a sloping aquifer, as first proposed by Boussinesq. Boussinesq proposed a modified version of Darcys law:

$$q_{GW}=-K_h*h*{\delta h}/{\delta x}*cos(\beta)-K_h*h*sin(\beta)$$

where q_{GW} is the groundwater flow to the stream, K_h is the hydraulic conductivity of the aquifer along the slope, h is the groundwater level above the aquifer base, β is the slope of the aquifer. For the meaning of variables see the sketch below:



When combined with the continuity equation, the so-called 'Boussinesq-Equation' is obtained:

$$\frac{\delta}{\delta x}* \left (h*\frac{\delta h}{\delta x} \right) * cos(\beta) + \frac{\delta h}{\delta x}*sin(\beta) = \frac{S_y}{K_h}*\frac{\delta h}{\delta t}$$

Brutsaerts simplification

Brutsaert (1994) linearized the equation:

$$q_{GW}=-\alpha*K_h*h_0*\frac{\delta h}{\delta x}*cos(\beta)-K_h*h*sin(\beta)$$

where α is a coefficient that equals more or less 0.5 and adjust for the use of h_0 instead of h.

The Boussinesq equation can be simplified to:

$$\left (\frac{\alpha*K_h*h_0*cos(\beta) }{S_y} \right) * \left (\frac{\delta^2(h_0-h)}{\delta x^2} \right) + \left (\frac{K_h*sin(\beta)}{S_y} \right) * \left (\frac{\delta*(h_0-h)}{\delta x} \right) = \frac{\delta h}{\delta t}$$

The shape parameter

There is a shape parameter that describes the ratio between advection and diffusion terms:

$$ J = \frac{\frac{K_h*sin \beta * X}{S_y}}{\frac{\alpha*K_h*h_0*cos \beta}{S_y}}=\frac{tan \beta * X}{\alpha * h_0}$$

A large J means steep slopw and shallow aquifer and a predomincance of advection, a small J means shallow slope and thick aquifer and a dominance of diffusion, for a slope of $\beta=0$ the solution approaches that of the Dupuit equation.

Solution for shallow slopes

For a shallow aquifer with $\beta = 0$ or the Dupuit conditions, the solution of the Boussinesq equation simplifies to

$$q_{GW}= \left (\frac{h^2_0*K_h}{X} \right) * \sum_{i=1}^{n}*exp \left [-\frac{(2*i-1)^2*\pi^2*K_h*h_0}{8*X^2*S_y}*t \right]$$

This equation represents a series of terms, the importance of terms for higher i diminishes rapidly. If only the first term is retained, we get

$$q_{GW}= \left (\frac{h^2_0*K_h}{X} \right) * exp \left [\frac{1.23*K_h*h_0}{X^2*S_y}*t \right]$$

this is the equation for the drainage of a Dupuit aquifer.

Solution for steep slopes

A general solution of the equation is given by Brutsaert(1994). For a steep aquifer and large J, some simplifications can be introduced:

$$B = h*S_y*X$$

where B is the gravity drainable water. In addition the outflow can be described by Darcy's law as:

$$q_{GW}=h*k_h*tan \beta$$

and hence (if no vertical recharge occurs:

$$q_{GW}=-\frac{dB}{dt}$$

After substitution this leads to

$$q_{GW}=K_h*h_0*tan \beta * exp \left (\frac{-K_h*tan \beta}{S_y * X}*t \right) $$

The equation can be used to estimate the outflow from a sloping (steep and shallow) aquifer as a function of hydraulic conductivity K_h, thickness of the aquifer h_0, slope of the aquifer $tan

\beta$, storativity S_y and length of the aquifer X as a function of time t. This equation can be simplified to

$$q_{GW}=a * h_0 * exp \left (\frac{-a}{S_y * X}*t \right) $$

where $a=K_h*tan \beta$.